

to a unification on a higher level. But the distinction mentioned above led to another most remarkable line of thought and research which tends more and more to govern mathematical doctrine. The methods of projection are based upon the motion or upon the transformation of figures. Under such a process some relations remain unaltered or invariant, others change. As analytical methods in the hands of Plücker and others began to accommodate themselves more closely to geometrical forms, as an intimate correspondence was introduced between the figure and the formula, it became natural to study the unalterable properties of the figure in the invariant elements of the formula. This is the origin and meaning of the doctrine of Invariants.¹ It is the great merit of the English school of mathematicians, headed by Boole, Cayley, and Sylvester, both to have first conceived the idea of a doctrine of invariant

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Invariants.

¹ "In any subject of inquiry there are certain entities, the mutual relations of which, under various conditions, it is desirable to ascertain. A certain combination of these entities may be found to have an unalterable value when the entities are submitted to certain processes or are made the subjects of certain operations. The theory of invariants in its widest scientific meaning determines these combinations, elucidates their properties, and expresses results when possible in terms of them. Many of the general principles of political science and economics can be expressed by means of invariative relations connecting the factors which enter as entities into the special problems. The great principle of chemical science which asserts that

when elementary or compound bodies combine with one another the total weight of the materials is unchanged, is another case in point. Again, in physics, a given mass of gas under the operation of varying pressure and temperature has the well-known invariant, pressure multiplied by volume and divided by absolute temperature. Examples might be multiplied. In mathematics the entities under examination may be arithmetical, algebraical, or geometrical; the processes to which they are subjected may be any of those which are met with in mathematical work. It is the principle which is valuable. It is the idea of invariance that pervades to-day all branches of mathematics" (Major P. A. MacMahon, Address, Brit. Assoc., 1901, p. 526).